Announcements

1) Assignment 4 up sometime tomorrow

2) Proof that every finite dimensional vector space over $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$ will be filled in, up on Canvas.

Philosophy: whenever possible, reduce proofs for finite dimensional vector spaces to $\mathbb{F}^n$ by quoting isomorphism.

Example 1: Define

$$T : \mathbb{R}^3 \to \mathbb{R}^2$$

$$T((x, y, z)) = (17x - 4y, 2z).$$

This is a linear map.

$$T((x,yz)) = (17x-4y, 2z)$$

Define the matrix

$$A = \begin{bmatrix} 17 & -4 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

If we think of

$(x,y,z) \in \mathbb{R}^3$ as a

column vector,

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 17 & -4 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 17x - 4y \\ 2z \end{bmatrix}$$

Which, converting back to a row, gives $T((x,y,z))$.

Theorem: (matrix form)

Let V and W be finite dimensional vector spaces over $\mathbb{F}$, with $n = \dim(V)$, $m = \dim(W)$.

Then if $T: V \to W$ is linear, $\exists$ a matrix

$$A \in M_{m \times n}(\mathbb{F})$$

such that T is given by the matrix A.

proof: We reduce to $V = \mathbb{F}^n$, $W = \mathbb{F}^m$.

Let $T: V \to W$ be linear.

Let $(f_k^n)_{k=1}^n$

and $(f_j^m)_{j=1}^m$ be

the standard bases of

$\mathbb{F}^n$ and $\mathbb{F}^m$, respectively.

$\forall\ k,\ 1 \le k \le n,$

$\exists$ scalars $(\alpha_{j,k})_{j=1}^{m}$

with

$$\boxed{\underbrace{\vec{T}(f_k^n)}_{\mathbb{F}^m} = \sum_{j=1}^{m} \alpha_{j,k} f_j^m}$$

Since $(f_j^m)_{j=1}^{m}$ is a basis

for $\mathbb{F}^m$.

Then if $x \in \mathbb{F}^n$,

we can write

$$x = \sum_{k=1}^{n} \beta_k f_k^n$$

for some $\beta_1, \beta_2, \ldots, \beta_n \in \mathbb{F}^n$.

By linearity of T,

$$Tx = T\left(\sum_{k=1}^{n} \beta_k f_k^n\right)$$

$$= \sum_{k=1}^{n} T(\beta_k f_k^n)$$

$$= \sum_{k=1}^{n} \beta_k T(f_k^n)$$

$$= \sum_{k=1}^{n} \beta_k \sum_{j=1}^{m} \alpha_{j,k} f_j^m$$

By definition of the matrix multiplication, if $\boxed{A = (\alpha_{j,k})_{j=1}^{m} {}_{k=1}^{n}}$, then

$$Ax = A \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=1}^{n} \alpha_{1,k} \beta_k \\ \vdots \\ \sum_{k=1}^{n} \alpha_{m,k} \beta_k \end{bmatrix}$$

Unraveling this in terms of the $(f_j^m)_{j=1}^m$ basis, we see that

$$Ax = Tx,$$

so T has the form of a matrix. $\square$

(interchange the order of summation in Tx)

Further Reduction

Any linear map between finite dimensional vector spaces can now be written as a matrix. Whenever convenient, we reduce to this case.

__Caution:__ Our matrix form was dependent on choosing a basis. In fact, it is even dependent on the order of a given basis. Changing the basis or order might change the matrix.

Example 2:

Let T be the linear map, $T: \mathbb{R}^2 \to \mathbb{R}^2$,

$$T((x,y)) = (3x-5y, -x+y).$$

Using our theorem, we can express T as the matrix

$$A = \begin{bmatrix} 3 & -5 \\ -1 & 1 \end{bmatrix}$$

with respect to the standard basis!

Let $\{b_1, b_2\}$ be the basis of $\mathbb{R}^2$ given by $b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$,

$$b_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$T(b_1) = (-2, 0)$

$T(b_2) = (8, -2)$

In the $\{b_1, b_2\}$ basis,

$$(-2, 0) = \alpha b_1 + \beta b_2$$
$$= \alpha(1,1) + \beta(1,-1)$$

If $\alpha = \beta = -1$, we have equality.

Similarly,

$$(8,-2) = \gamma b_1 + \delta b_2$$

$$= \gamma(1,1) + \delta(1,-1)$$

$\delta = 5$, $\gamma = 3$

are the desired coefficients.

$$T(b_1) = -b_1 - b_2$$

$$T(b_2) = 3b_1 + 5b_2$$

With respect to the $\{b_1, b_2\}$ basis,

T has the matrix

$$\boxed{\begin{bmatrix} -1 & 3 \\ -1 & 5 \end{bmatrix}}$$

Change of Basis

If $\{x_1, x_2, \ldots, x_n\}$

and $\{y_1, y_2, \ldots, y_n\}$

are two bases of V

over $\mathbb{F}$, $\exists$ a linear

transformation, called the

Change of basis map

induced by the association

$$x_i \longmapsto y_i$$

and then extending by linearity.